Advances in Mathematics of Communications
Volume 5, No. 2, 2011, 157–160
doi:10.3934/amc.2011.5.157

A GEOMETRIC PROOF OF THE UPPER BOUND ON THE SIZE OF PARTIAL SPREADS IN $H(4n+1, q^2)$

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(Communicated by Ivan Landjev)

Abstract. We give a geometric proof of the upper bound of $q^{2n+1}+1$ on the size of partial spreads in the polar space $H(4n+1, q^2)$. This bound is tight and has already been proved in an algebraic way. Our alternative proof also yields a characterization of the partial spreads of maximum size in $H(4n+1, q^2)$.

1. Introduction

A classical finite polar space is an incidence structure, consisting of the totally isotropic subspaces of a projective space with respect to a non-degenerate sesquilinear form or a non-degenerate quadratic form. All dimensions will be assumed to be projective from now on, and we will also refer to m-dimensional subspaces as simply m-spaces. In particular, the 0- and 1-dimensional subspaces of such a polar space are known as its *points* and *lines*, respectively. The *generators* are its subspaces of maximal dimension. A *partial spread* of a classical finite polar space is a set of generators with no two incident with a common point. If a partial spread actually partitions the point set of the polar space, it is said to be a *spread*.

The Hermitian variety $H(n, q^2)$ is a particular type of classical finite polar space, consisting of the subspaces in $\mathsf{PG}(n, q^2)$, the points of which all have homogeneous coordinates $(x_0, \ldots, x_n)$ satisfying the equation $x_0^{q+1} + \ldots + x_n^{q+1} = 0$. In this polar space, the generators are $(n-1)/2$-dimensional, if n is odd, or $(n-2)/2$-dimensional, if n is even, and the number of points is given by $|H(n, q^2)| = (q^{n+1} + (-1)^n)(q^n - (-1)^n)/(q^2-1)$. We refer to [4] for proofs and much more information on Hermitian varieties and polar spaces in general.

Thas [6] proved that in $H(2n+1, q^2)$ spreads, or thus partial spreads of size $q^{2n+1}+1$, cannot exist, which has made the question on the size of a partial spread in such a polar space, an intriguing question. Improved upper bounds on the size of partial spreads in $H(2n+1, q^2)$ were proved in [2].

On the other hand, partial spreads of size $q^{n+1}+1$ in $H(2n+1, q^2)$ were constructed for all $n \geq 1$ in [1], by use of a symplectic polarity of the projective space $\mathsf{PG}(2n+1, q^2)$, commuting with the associated Hermitian polarity. In the Baer subgeometry of points on which these two polarities coincide, a (regular) spread of the induced symplectic polar space $W(2n+1, q)$ can always be found, and these

2000 *Mathematics Subject Classification:* 05B25, 51E23.
Key words and phrases: Hermitian varieties, partial spreads.
This research is supported by the Research Foundation Flanders-Belgium (FWO-Vlaanderen).

$q^{n+1}+1$ generators extend to pairwise disjoint generators of $H(2n+1,q^2)$. Maximality of partial spreads of $H(2n+1,q^2)$ constructed in this way was also shown for $n=1,2$ in [1] and for all even n in [5].

In [3], De Beule and Metsch proved that the maximum size of a partial spread in $H(5,q^2)$ is q^3+1, and they also obtained additional information on partial spreads meeting that tight bound. In particular, they found that every generator of $H(5,q^2)$, not meeting any element of such a partial spread S in a line or more, meets exactly q^2-q+1 elements of S in a point.

Using techniques from algebraic graph theory, we recently proved in [7] that the size of a partial spread in $H(4n+1,q^2)$ is at most $q^{2n+1}+1$, and this bound is thus tight as well. It turns out that a geometric property of partial spreads of maximum size in $H(5,q^2)$ can be generalized, and in fact paves the way for a new, completely geometric proof of the upper bound in $H(4n+1,q^2)$.

2. Tools

We first state a lemma by Thas [6].

Lemma 2.1. *Let π_1,π_2 and π be three mutually disjoint generators in $H(2n+1,q^2)$. The set of points on π_1, that are on a (necessarily unique) line of $H(2n+1,q^2)$ meeting both π and π_2, form a non-singular Hermitian variety in π_1.*

Corollary 2.2. *Let π_1,π_2 and π be three mutually disjoint generators in $H(2n+1,q^2)$. The number of generators meeting π in an $(n-1)$-space, and meeting both π_1 and π_2 in a point is $|H(n,q^2)|=\frac{(q^{n+1}+(-1)^n)(q^n-(-1)^n)}{q^2-1}$.*

Proof. We let $\perp$ denote the Hermitian polarity of $\mathsf{PG}(2n+1,q^2)$, associated with the polar space. It is obvious that every generator meeting π in an $(n-1)$-space, can meet π_1 and π_2 in at most one point. On the other hand, through any point $p_1\in\pi_1$, there is a unique generator $\langle p_1,p_1^\perp\cap\pi\rangle$ meeting π in an $(n-1)$-space. Hence we have to determine the number of points $p_1\in\pi_1$ such that the generator $\langle p_1,p_1^\perp\cap\pi\rangle$ also meets π_2 in a point.

First suppose that a point $p_1\in\pi_1$ is such that the generator $\langle p_1,p_1^\perp\cap\pi\rangle$ meets π_2 in a point p_2. In that case, the line p_1p_2 is a line of $H(2n+1,q^2)$, meeting π as well, as $p_1^\perp\cap\pi$ is a hyperplane of $\langle p_1,p_1^\perp\cap\pi\rangle$. Conversely, suppose a point $p_1\in\pi_1$ is on a line of $H(2n+1,q^2)$, meeting π in p and π_2 in p_2. In that case, both p_1 and p are in the generator $\langle p_1,p_1^\perp\cap\pi\rangle$, and hence so is the entire line p_1p, including the point p_2. The desired result thus follows from Lemma 2.1. $\square$

3. The proof

Theorem 3.1. *The size of a partial spread S in $H(4n+1,q^2), n\geq 1$, is at most $q^{2n+1}+1$. If $|S|>1$ and $\pi\in S$, then every generator meeting π in a $(2n-1)$-space, will meet the same number of other elements of S in just a point, if and only if $|S|=q^{2n+1}+1$. In that case, that number must be q^{2n}.*

Proof. Let S be a partial spread of size at least 2 in $H(4n+1,q^2)$. Consider a fixed element $\pi\in S$. Let $\{N_i|i\in I\}$ be the set of generators meeting π in a $(2n-1)$-space. As the number of $(2n-1)$-spaces in a generator equals $(q^{4n+2}-1)/(q^2-1)$, and the number of generators through any $(2n-1)$-space in $H(4n+1,q^2)$ is given by $q+1$, the cardinality of I is $\frac{q^{4n+2}-1}{q^2-1}q$.

Note that any generator N_i and any generator in $S\backslash\{\pi\}$, are either disjoint or meet in a point. For every $N_i, i \in I$, let t_i denote the number of generators in $S\backslash\{\pi\}$, meeting N_i in a point. We now count the number of pairs (N_i, π'), with π' an element of $S\backslash\{\pi\}$ meeting N_i in a point, in two ways. As through every point p' on an element π' of $S\backslash\{\pi\}$, there is a unique generator meeting π in a $(2n-1)$-space, we obtain:

$$\sum_{i\in I} t_i = (|S|-1)\frac{q^{4n+2}-1}{q^2-1}. \tag{1}$$

Now we count the number of ordered triples (N_i, π_1, π_2), with π_1 and π_2 two distinct elements of $S\backslash\{\pi\}$, both meeting N_i in a point. We know from Corollary 2.2 that for every two distinct elements of $S\backslash\{\pi\}$, there will be exactly $|H(2n, q^2)|$ generators N_i, meeting both of them in a point. Hence we obtain:

$$\sum_{i\in I} t_i(t_i-1) = (|S|-1)(|S|-2)\frac{(q^{2n+1}+1)(q^{2n}-1)}{q^2-1}. \tag{2}$$

Combining (1) and (2), we find:

$$\sum_{i\in I} t_i^2 = (|S|-1)\frac{q^{2n+1}+1}{q^2-1}\Big((q^{2n+1}-1)+(|S|-2)(q^{2n}-1)\Big). \tag{3}$$

As $(\sum_{i\in I} t_i)^2 \leq (\sum_{i\in I} t_i^2)|I|$, with equality if and only if all t_i are equal, this implies:

$$(|S|-1)^2\left(\frac{q^{4n+2}-1}{q^2-1}\right)^2 \leq (|S|-1)\frac{q^{2n+1}+1}{q^2-1}\Big((q^{2n+1}-1)+(|S|-2)(q^{2n}-1)\Big)\frac{q^{4n+2}-1}{q^2-1}q,$$

with equality if and only if all t_i are equal. Since we assumed that $|S| > 1$, we can cancel factors on both sides to obtain:

$$(|S|-1)(q^{2n+1}-1) \leq \Big((q^{2n+1}-1)+(|S|-2)(q^{2n}-1)\Big)q,$$

implying that $|S| \leq q^{2n+1}+1$, with equality if and only if all t_i are equal. In that case, their constant value must equal $(\sum_{i\in I} t_i)/|I| = q^{2n}$.

□

4. Remark

This technique fails when applied to partial spreads in $H(4n+3, q^2)$, where it yields a negative lower bound on the size instead.

Acknowledgements

The author is grateful to Jan De Beule for his helpful suggestions while preparing this manuscript.

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Received March 2010; revised August 2010.

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